Pennsylvania State University

University of Georgia

MAC-CPTM Situations Project

Situation 20: Area of Plane Figures

Prompt

A teacher in a geometry class introduces formulas for the areas of parallelograms, trapezoids, and rhombi. She removes the formulas from the overhead and poses several area problems to her students. One student volunteers the correct answers very quickly. Another student asks, "How did you memorize the formulas so fast?" The first student responds, "I didn't memorize the formulas. I can just see what the area should be."

Commentary

The four foci for this situation reflect relationships among three classes of figures: parallelograms, trapezoids, and rhombi. In Foci 1 and 2, strategies for a class of figures are applied to one of its subclasses. The two foci differ in terms of whether the strategy is the application of a known formula or the application of a method that develops the known formulas. Foci 3 and 4 involve decomposition of quadrilaterals, with the former emphasizing efficient calculation and the latter targeting the logical development of mathematics from what is known to what is needed.

Mathematical Foci

Mathematical Focus 1

Because rhombi, squares, and rectangles are parallelograms, the area for each can be found by $A = bh$, *where* b *represents the base and* h *represents the height.*

A parallelogram is a quadrilateral with two pairs of parallel sides; rhombi, squares and rectangles are special parallelograms. The Venn diagram in Figure 1 represents these relationships. One could conclude that the areas of these parallelograms would be found in the same way, the product of the length of a base and the corresponding height.

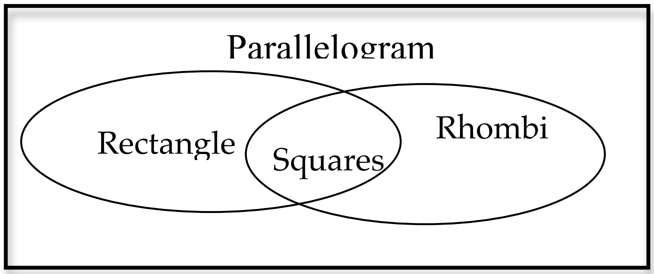


Figure 1

Mathematical Focus 2

Any parallelogram, trapezoid or rhombus can be decomposed into two triangles, the sum of whose areas is the area of the original quadrilateral—which suggests a way to generate familiar area formulas.

The method of constructing a diagonal to decompose a parallelogram, rhombus, or trapezoid into two triangles elucidates possible derivations of the area formulas. For example, in Figure 2, diagonal $\overline{AC}$ divides parallelogram *ABCD* into two congruent triangles, $\Delta ABC \cong \Delta CDA$. Using the base and height of the parallelogram, the area of each triangle is $A_{triangle} = \frac{1}{2}bh$. Since the area of parallelogram *ABCD* is equal to the sum of the areas of the two triangles,

$$A_{parallelogram} = \frac{1}{2}bh + \frac{1}{2}bh = 2\left(\frac{1}{2}bh\right) = bh$$

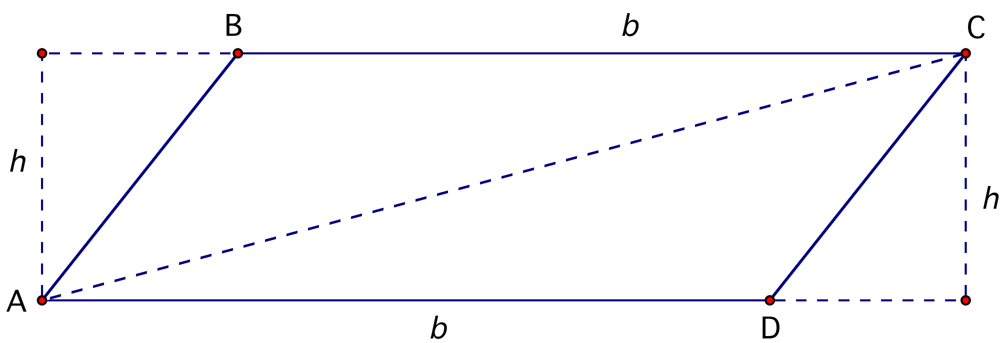


Figure 2

Similarly, diagonal $\overline{PS}$ of trapezoid *PRST* in Figure 3 subdivides the trapezoid into two triangles, ΔPRS and ΔPST. In general, the two triangles are not congruent but their areas sum to the area of the trapezoid. Using one base and the height of the trapezoid, the area of ΔPRS is $A = \frac{1}{2}ah$, and the area of ΔPST is

$A = \frac{1}{2}bh$. The area of trapezoid *PRST* is equal to the sum of the areas of ΔPRS and ΔPST: $A = \frac{1}{2}ah + \frac{1}{2}bh = \frac{1}{2}h(a+b)$.

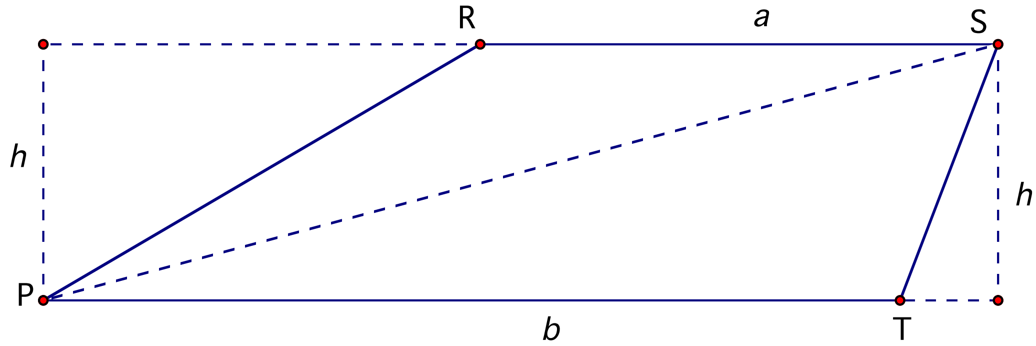


Figure 3

In the case of the rhombus, diagonal $\overline{XZ}$ subdivides rhombus *WXYZ* into two congruent triangles, ΔWXZ and ΔYZX, as shown in Figure 4. Let *V* be the intersection of $\overline{XZ}$ and $\overline{WY}$, which are diagonals of a rhombus and thus perpendicular bisectors of each other. By defining diagonal $\overline{XZ}$, which has length d_1, to be the base of both ΔWXZ and ΔYZX, then segment $\overline{WV}$ with length $\frac{1}{2}d_2$ would be the altitude of ΔWXZ. Given $\overline{WY}$ a diagonal of length d_2, segment $\overline{VY}$ with length $\frac{1}{2}d_2$ would be the altitude of ΔYZX. The area of each triangle, ΔYZX and ΔWXZ, would be $A_{triangle} = \frac{1}{2}d_1\left(\frac{1}{2}d_2\right) = \frac{1}{4}d_1d_2$, and thus the area of rhombus *WXYZ* would be: $A_{rhombus} = \frac{1}{4}d_1d_2 + \frac{1}{4}d_1d_2 = 2\left(\frac{1}{4}d_1d_2\right) = \frac{1}{2}d_1d_2$.

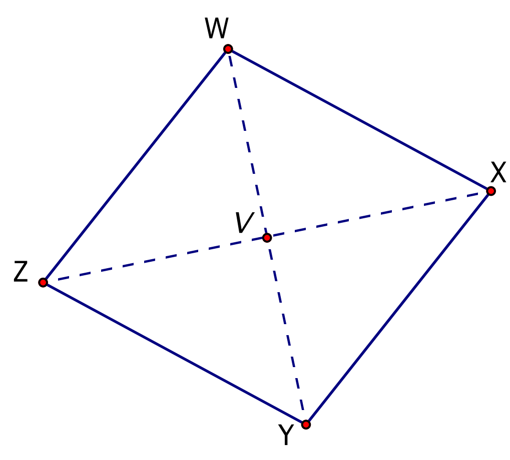


Figure 4

It should also be noted that, because a rhombus is a parallelogram (see discussion of Mathematical Focus 1), it follows from the discussion of parallelogram *ABCD* above that a rhombus can be decomposed into two triangles whose sum is equal to the area of the rhombus.

Mathematical Focus 3

A parallelogram, trapezoid, or rhombus can be decomposed into a combination of polygons, the sum of whose areas can be calculated efficiently, and that choice depends both on the original figure and the measures involved.

How we decompose a quadrilateral to determine its area depends on the specific type of quadrilateral we have. Some decompositions are not possible. For example, one way to decompose a stereotypical scalene trapezoid like that in Figure 5 involves two triangles and a rectangle. In contrast, it makes no sense to use two triangles and a rectangle in a decomposition of a right trapezoid such as ABCD in Figure 6.

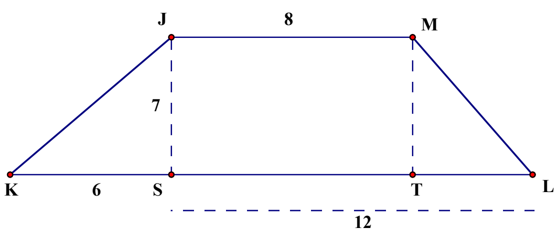


Figure 5

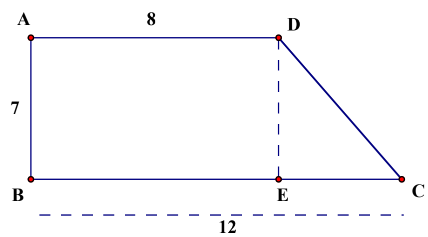


Figure 6

In some cases, the general nature of the decomposition might be the same but the calculation can be done more efficiently based on observations about the numbers involved. For example, the decomposition of a stereotypical trapezoid into one rectangle and two triangles works with an isosceles trapezoid. To determine the area of the isosceles trapezoid JKLM in Figure 7, we can decompose the figure into rectangle JSTM and congruent triangles JKS and MTL. The calculation is slightly easier if we recognize the equal areas of the two right triangles: Area trapezoid JKLM is $\boxed{2(1/2}\ 1/2\ (12-8))(7) + (8)(7) = \boxed{1}(2)(7) + 8(7) = 10(7) = 70$ square units.

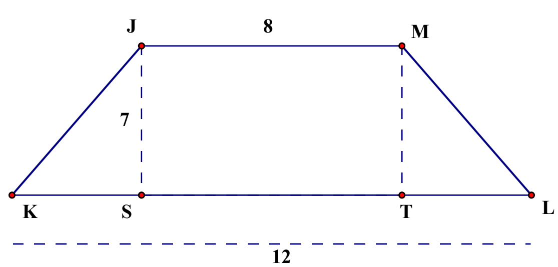
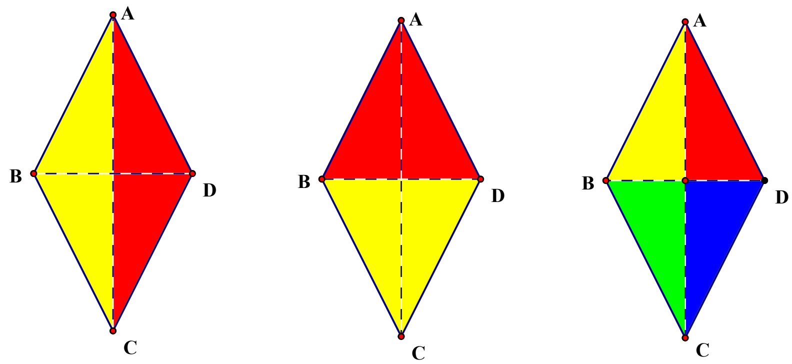


Figure 7

For figures of the same type, we might use slightly different computations or decompositions depending on the numbers involved, particularly in the absence of convenient formulas. For example, to determine the areas of the rhombus in Figure 8, thinking in terms of two triangles or four triangles yields easy calculations. The slight change in one measure creates the rhombus in Figure 9, for which only one of the three options seems to produce a slightly easier mental calculation.

BD = 4
AC = 8



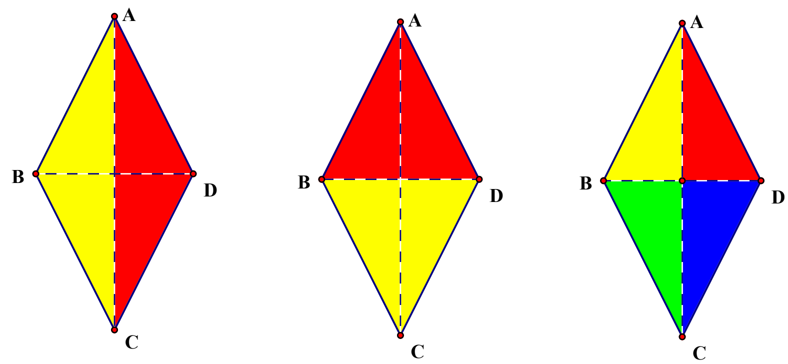
Twice 1/2 (8) (1/2 4) or 2 4 (2), 16 square units

Twice 1/2 (4) (1/2 8) or 2 2 (4), 16 square units

Four times 1/2 (1/2 8) (1/2 4) or 2 (4)(2), 16 square units

Figure 8

BD = 3
AC = 8



Twice 1/2 (8) (1/2 3) or 2 4 (1.5), 12 square units

Twice 1/2 (3) (1/2 8) or 3 (4), 12 square units

Four times 1/2 (1/2 8) (1/2 3) or 2 (4)(1.5), 12 square units

Figure 9

Mathematical Focus 4

Areas of rectangles, parallelograms, and triangles and trapezoids can be logically developed in that order.

A deductive approach to the development of area formulas is consistent with the systematic nature of Euclidean geometry. Covering a rectangular region with an array of unit squares illustrates the multiplicative nature of the area formula, $A = lw$. The formula for the area of a rectangle can be used to develop the formula for the area of a parallelogram. An altitude from vertex A in parallelogram ABCD creates right ΔAED. Translating ΔAED along a vector from

A to B creates rectangle ABFE whose area is congruent to the area of parallelogram ABCD, as in Figure 10. Thus, $A_{ABCD} = bh$.

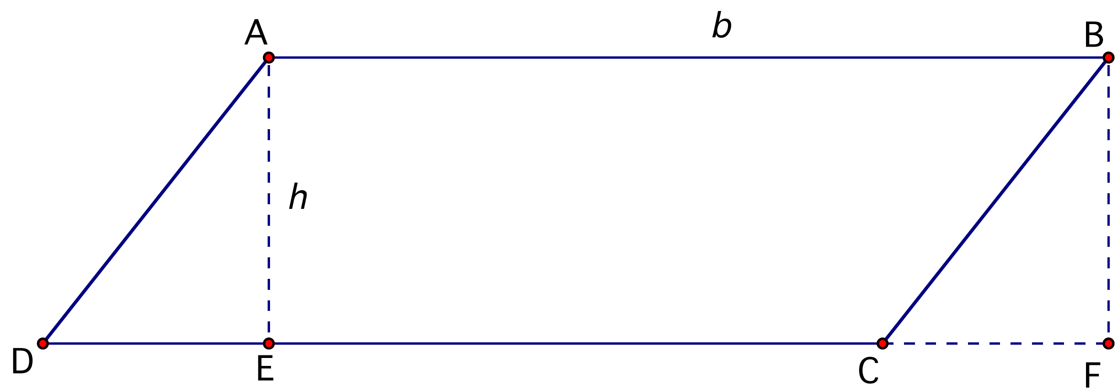


Figure 10

The area formula for a parallelogram can be used to find the formulas for the area of a triangle and the area of a trapezoid. Both cases involve rotating a polygon 180° about the midpoint of one side. Figure 11 shows how the 180° rotation of ΔABC about midpoint M, which creates quadrilateral AC'BC with opposite sides congruent. So, AC'BC is a parallelogram whose area is twice the area of original triangle; thus $A_{AC'BC} = bh$ implies $A_{\Delta ABC} = \frac{1}{2}bh$.

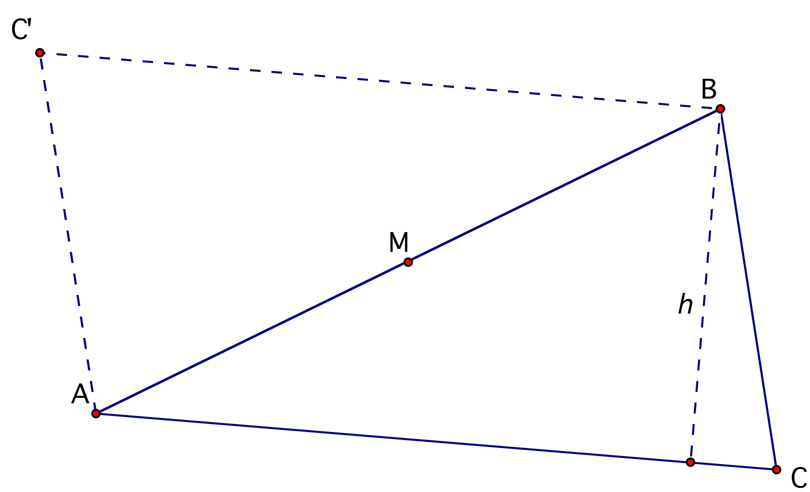


Figure 11

Similarly, Figure 12 illustrates how rotating trapezoid ABCD 180° about midpoint *M* of a non-parallel side creates quadrilateral C'D'CD with opposite angles congruent. C'D'CD is a parallelogram whose area is twice the area of the original trapezoid; thus $A_{C'D'CD} = h(a+b)$ implies $A_{ABCD} = 1/2\ h(a+b)$.

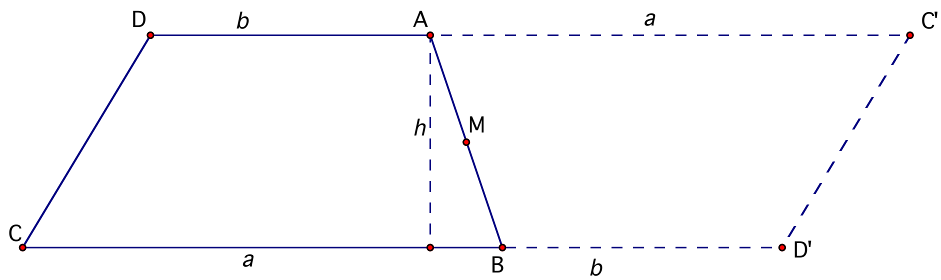


Figure 12

Post Commentary

Foci 1, 2 and 4 involve the conservation of area under translations and rotations. These transformations are not as apparent in Focus 3, though one can interpret the calculation string for the area of the isosceles trapezoid as $\boxed{2(1/2}$ 1/2(12-8))(7) is the area of two congruent triangles and the corresponding term, $\boxed{1}$ 1/2(12-8))(7), is the area of the rectangle created by the pairing of one triangle and the image of the second triangle after a translation and reflection are consecutively applied, as illustrated in Figure 1.

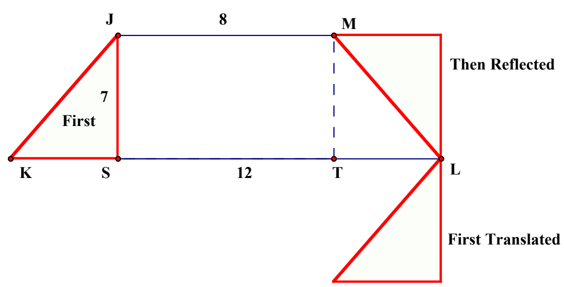


Figure 1

All of the foci have implications for interpreting area formulas and related expressions. For example, Focus 1 suggests we would see $A=lw$ as a product of base length l and height w. In Focus 2, the expressions are expanded or simplified versions of the sum of the areas of two triangles. The numerical expressions in Focus 3 are manipulated for ease of calculation. Focus 4 involves seeing formula as derivations or sources of other area formulas.